Now say that X is a set, $\mathcal{R}$ is a ring of subsets and we have a function $\mu : \mathcal{R} \to [0, \infty)$. This is the "measure", what we're looking for. One of the properties we need for this function is *additivity*:

$$\mu(A \cup B) = \mu(A) + \mu(B), \qquad \text{if } A, B \in \mathcal{R}, A \cap B = \emptyset$$

From just this property we can derive a number of properties about μ.

Simple Properties of Measure

1. $\mu(\emptyset) = 0$ $[\emptyset = A \ominus A]$

   Proof:
   $$\mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset) = \mu(A)$$

2. *Monotonicity* If $A, B \in \mathcal{R}$, $A \subset B$ then $\mu(A) \leq \mu(B)$.

   Proof:
   $$B = A \cup (B \ominus A) \Rightarrow \mu(B) = \mu(A) + \mu(B \ominus A) \geq \mu(A)$$

   So monotonicity is a consequence of additivity and positivity.

3. *Finite Additivity* If $A_1, \ldots, A_n \in \mathcal{R}$ are pairwise disjoint then
   $$\mu\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu(A_i).$$

   Proof: Use induction on N. Assume that this statement is true for $N - 1$ disjoint sets. Then look at $A_N, B = \bigcup^{N-1} A_i \in \mathcal{R}$ and A_N and B are disjoint, so
   $$\mu(A_N \cup B) = \mu(A_N) + \mu(B) \Rightarrow \mu(A_N) + \sum_{i=1}^{N-1} \mu(A_i) = \sum_{i=1}^{N} \mu(A_i)$$

   by the inductive hypothesis and we are done.

4. *Lattice Property* For $A, B \in \mathcal{R}$
   $$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

   Proof: To prove this we break the sets up into disjoint unions and reassemble.
   $$A = (A \backslash B) \cup (A \cap B), \qquad B = (B \backslash A) \cup (A \cap B)$$

   and
   $$A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$$

   we can easily see from this breakdown that
   $$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

5. *Finite Sub-additivity* For $A_i \in \mathcal{R}, i = 1, \ldots, N$

$$\mu\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} \mu(A_i)$$

Proof: Looks exactly the same as the finite additivity proof.

Definition. Countable Sub-additivity A measure on a ring $\mathcal{R}$ is said to be **countably sub-additive** if given a countable collection of $A_i \subset \mathcal{R}$, $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup A_i \in \mathcal{R}$ if we then have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

(note that this makes sense even if the right diverges).

Theorem. *Measures*

1. *Our length measure is countably subadditive.(Once we have shown this it follows that the measure is countably additive, and thus technically a measure)*

2. *(Cartheodory) We can extend our current measure to a countably additive measure. This is "Lebesgue" measure.*

Theorem. *If $A_i \in \mathcal{R}$, $i = 1, \ldots,$ and $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$ by assumption, and $A_i \cap A_j = \emptyset$ Then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i), \qquad A_i \cap A_j = \emptyset$$

Proof. Let $\bigcup^{N} A_i \in \mathcal{R}$ and

$$\mu(\bigcup^{N} A_i) = \sum_{i=1}^{N} \mu(A_i)$$

and $\bigcup^{N} A_i \subset A$ implies that

$$\boxed{\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \mu\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu(A_i) \forall N \implies \mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)}$$

□

this coupled with subadditivity means that

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$$

and this is what it means to be a measure.

Idea: Caratheodory Try to replace $\mathcal{R}$ by the biggest possible collection of subsets on which μ (extends to be) countably additive

We want to extend $\mathcal{R}$ so that $\bigcup^{\infty} A_i \in \mathcal{R}$, so its not just an assumption. This is the core of measure theory.

Example. "length" in $\mathcal{R}$ for today we refer to "volume" in $\mathbb{R}^n$. $I \subset \mathcal{R}^n$ is a multi-interval

$$I = I_1 \times I_2 \times \cdots \times I_n$$

then

$$\mu_L(I) = \prod_{i=1}^{n} (b_i - a_i)$$

now $\mathcal{R}_L$ is the finite unions of disjoint multi-intervals so the volume in $\mathcal{R}_L$ is

$$\mu_L = \sum_{k=1}^{N} \mu(I^{(k)}), \qquad A = \bigcup_{k=1}^{N} I^{(k)}$$

But what about countable additivity? when all the multi-intervals can be small and nasty?

$\square$